Praktikum aus numerischer Astronomie
Solving ordinary differential equations

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Zusammenfassung

In this practical an easy differential equation was solved in four different ways. An error estimation was done for the Euler forward- resp. backward-formula and the Runge-Kutta method.

1 Solutions

1.1 Analytic Solutions

Because of the linearity of $f(x,y) = 100(x-y)+1$, the differential equation $\frac{dy}{dx} = f(x,y)$ can be solved analytically. The Euler method and the Runge-Kutta method reduces each to a first order finite difference equation. With step size $h = \frac{\kappa}{100}$, $\kappa \leq 1$, $x_n = hn$ and initial condition $y(0) = 1$, thus $y_0 = 1$, the explicit terms are

$$y_n = hn + \lambda^n$$

with

$$\lambda = \begin{cases} 1-\kappa & \text{Euler forward} \\ \frac{1}{1+\kappa} & \text{Euler backward} \\ 1-\kappa+\frac{\kappa^2}{2}-\frac{\kappa^3}{6}+\frac{\kappa^4}{24} & \text{Runge-Kutta} \end{cases}$$

The error ϵ can be expressed therefore in the form

$$\epsilon = \left|\lambda^n - e^{-\kappa n}\right|$$

The Adams-Moulton-Bashford predictor-corrector scheme reduces for $h = \frac{1}{100}$ to the recursion

$$7200y_{n+3} - 5150y_{n+2} + 3400y_{n+1} - 1250y_n = 42n + 147$$

with initial condition $y_0 = 1$, $y_1 = \frac{77}{200}$ and $y_2 = \frac{257}{1600}$. At least theoretically this can be solved explicitly:

$$y_n = A\alpha^n + B\beta^n \cos(\varphi n) + C\beta^n \sin(\varphi n) + Dn + E$$

with $\alpha \approx 0.482$, $\beta \approx 0.600$, $\varphi \approx 1.375$ und A, B, C, D, E appropriate.

1.2 Numerical solutions

Numerical solutions in $x \in [0, 0.2]$ for all cases are shown in Abbildung 1.

1.3 Error estimation

For the Euler-forward, Euler-backward and Runge-Kutta method, the biggest κ was evaluated, so that

$$\text{MaxError}(\kappa) = \max |y_n(\kappa) - y(x_n)| \leq 0.005$$

The number of required steps to cover the whole interval is approximately the reverse of h, thus the reverse of κ multiplied by 100.

method	maximal κ	# of steps
Euler forward	0.026877	3721
Euler backward	0.027494	3638
Runge-Kutta	0.929785	108

The alternative approach as proposed in the description is less feasible since *maxerr* is quite large and therfore h_{min} becomes smaller too fast.

1.4 Variable step size

The required step size to achieve 0.5% accuracy is growing with growing x and settles down at approximately $\kappa = 2.7853$. In this scenario there are 39 integration steps required. In Abbildung 2 the correlation $h_i(x_i)$ together with the analytic solution is illustrated.

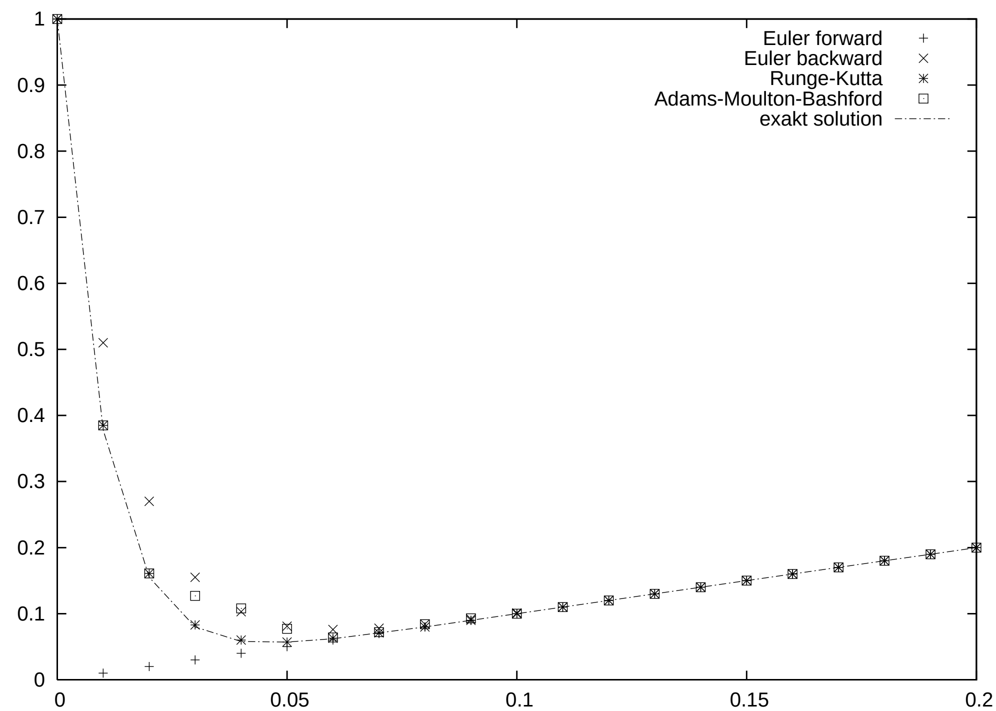

Abbildung 1: Numerical solutions

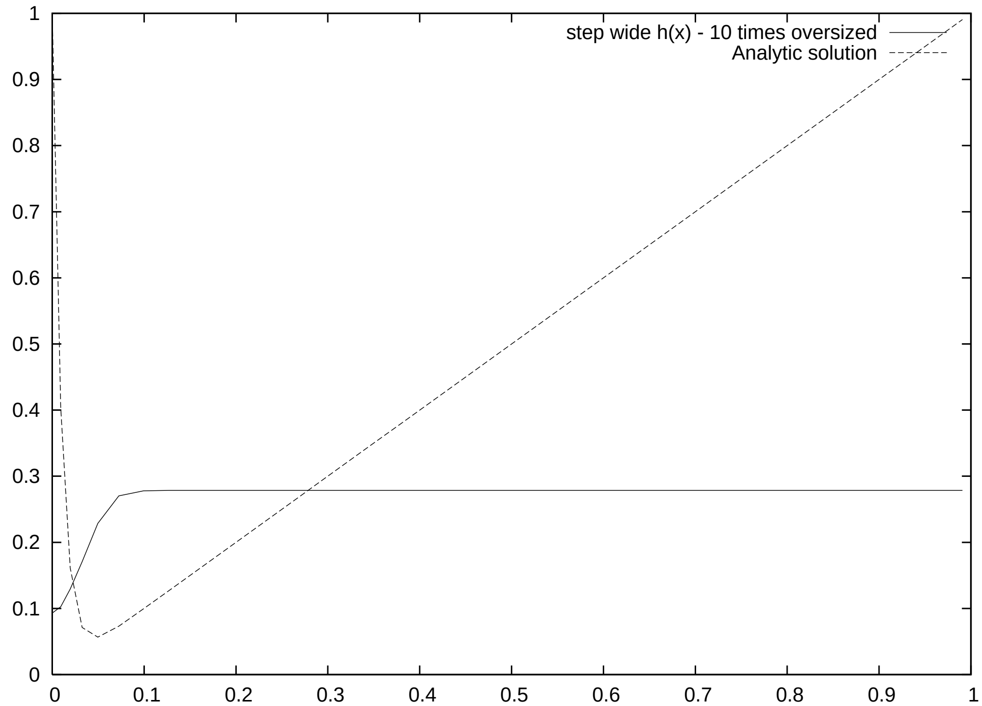

Abbildung 2: step wide $h(x) \times 10$ to guarantee accuracy of $\leq 0.5\%$ (lower graph) and exact analytic solution